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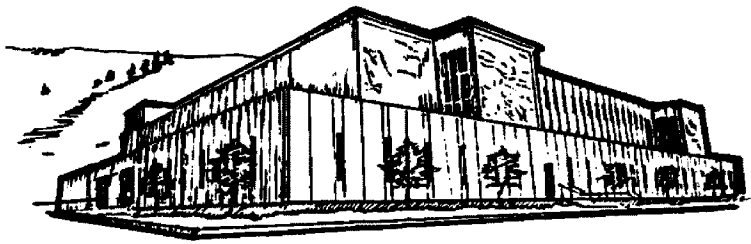
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Luc Desclotres

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THE CONSIDERATION OF SITE DETERIORATION  
IN THE ECONOMICS OF EVEN-AGED FORESTRY

By

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Presented in partial fulfillment of the requirements

for the degree of

Master of Science

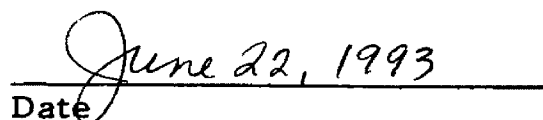
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**The Consideration of Site Deterioration  
in the Economics of Even-Aged Forestry (59 pp.)**

Director: David H. Jackson



The conventional approach to optimum investment in even-aged forestry under conditions of perfect competition assumes that site quality is constant over time. When site deterioration occurs, some additional techniques are needed to study the incidence of decreasing site quality on the investment strategies in even-aged forestry. A variation of the model proposed by Jackson, in which establishment input is a decision variable along with rotation age, is used. Timber yield is expressed by the product of Schumacher's yield function for normal stands and a stocking function which gives the percentage of normality over time. The site index is incorporated into the model to account for site quality. When site deterioration is exogenous, caused by external forces, comparative statics indicates that establishment input must be decreased and rotation age must be increased. When site deterioration is endogenous, resulting from silvicultural practices, the planning horizon is cut back by exhaustion. User-cost analysis shows that both establishment input and rotation age must be decreased as exhaustion approaches.

*To Michel, with whom it all begins.*

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## CHAPTER ONE:

### INTRODUCTION

The conventional approach to optimum investment in even-aged forestry under conditions of perfect competition is based on the assumption that the forester wants to maximize the land owner's income (rent) or wealth (land value). Rent and land value are two different *objective functions* that reach their maxima in the same conditions, since land value is nothing but capitalized rent in modern capital theory. Therefore, the choice of the objective function between rent and land value is not a matter of importance.

The fundamental problem in the economics of even-aged forestry is to determine when to harvest the stand so as to maximize rent and land value. The objective function is a mathematical expression in which timber yield is the main variable, and the unit price of timber and the interest rate are the main parameters. Timber yield itself depends on the age of the stand at the time of harvest, or rotation age<sup>1</sup>. At any time, the forester may choose to let trees grow longer or to liquidate timber and start a new rotation. Rotation age is therefore the *primary* decision variable of the objective function.

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<sup>1</sup>The rotation is the length of the growth period from stand establishment to harvest in even-aged forestry.

In addition, timber yield is largely related to the intensity of stand management. The forester can accelerate tree growth through site preparation, planting, and tending. Establishment input, which measures the effort spent on the stand at inception, is sometimes used as a *secondary* decision variable of the objective function. Intermediate inputs that take place during the rotation have also some influence on the yield of timber. They could be treated as additional decision variables of the objective function but are disregarded in most economic models for simplicity.

Evidently, timber yield is also tied to site quality. In even-aged forestry, site quality is measured by the site index, that is, the average height of the dominant trees in the stand at some specified reference age. The conventional approach to optimum investment in even-aged forestry assumes that ecological as well as economic conditions are permanent. In particular, site quality is taken as constant over time; the site index is treated as a fixed parameter which defines forever the ecological conditions of rent and land-value maximization. Because the site index and all other parameters of the objective function are fixed, the optimal values of rotation age and establishment input are the same for all rotations. More importantly, the conventional approach assumes that tree growth is perpetual. Site quality is and stays at a level where trees can regenerate. Thus the number of rotations is infinite.

Nevertheless, there is evidence that the site deteriorates on some forest

lands. Site deterioration may be *exogenous* if it is caused by external forces or *endogenous* if it results from silvicultural practices. In any case, the conventional approach to optimum investment in even-aged forestry does not allow for site deterioration. This study proposes a different approach in which the assumption of permanent conditions is relaxed. When site deterioration is exogenous, the site index is still treated as a parameter of the yield function but is allowed to decrease from one rotation to another. When site deterioration is endogenous, the assumption of perpetual growth is removed and the site index is treated as a decreasing function of cumulative yield. After a number of rotations, the site index reaches some bottom value at which trees of the desired species do not regenerate.

When ecological or economic conditions vary, and in particular when the site deteriorates, the optimal values of rotation age and establishment input must change from one rotation to another. The objective of this study is to determine in which direction the decision variables should progress to maximize rent and land value with exogenous or endogenous site deterioration. First, site quality and its variable nature will be discussed (Chapter 2). Next, the various models used by the conventional approach to optimum investment in even-aged forestry will be presented (Chapter 3). A new model will be proposed afterward, and comparative statics will be used to study the effects of exogenous site deterioration on the model (Chapter 4). Then the concept of user cost will be introduced to study the effects of

endogenous site deterioration (Chapter 5). Finally, a conclusion will be reached on how site deterioration should affect the investment strategies in even-aged forestry under conditions of perfect competition (Chapter 6).

## CHAPTER TWO:

### SITE QUALITY AND ITS CHANGE OVER TIME

In forestry, site is an integrative concept that encompasses all climatic and edaphic<sup>1</sup> factors affecting a particular piece of land (Bruce and Schumacher 1950). The effect of all these factors on tree growth determines the productive capacity of the site and is commonly called site quality. There has been a lot of controversy surrounding the actual measurement of site quality, but the average height of the dominant and codominant trees has emerged as the most practical tally of site quality for even-aged stands (Frothingham 1921, Mader 1963). Different levels of site quality correspond to different height development patterns of dominant trees, which can be represented by different height-age curves. Theoretically, it is possible to draw a whole family of height-age curves with each curve corresponding to one level of site quality. The height in feet attained at some specified reference age on each curve is called the *site index*; it is a numerical expression of site quality for even-aged stands (Clutter and others 1983).

The concept of site used by foresters is very similar to that of environment used by ecologists. The environment includes time as a dimension by which climatic, edaphic, and other habitat factors are qualified (Billings 1952). On

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<sup>1</sup>Edaphic factors are environmental factors pertaining to soil.

the other hand, the time dimension of the site is generally ignored in forestry. Still, there is no reason to believe that the site does not change over time while the environment does. The site may alter, even within a few rotations. There is only one height development pattern of dominant trees per rotation, so the site index remains constant *for the duration of a rotation*. However, a variation in site quality would result in distinct height development patterns and hence different site indices *from one rotation to another*.

Climatic factors can be assumed to be fairly constant within the scope of man's socio-economic timeframe. The edaphic component of site quality is the most likely to change within a few rotations. The variability of soil properties over time and its importance have long been recognized in agriculture. Bunce (1942) made a distinction between *fertility depletion* and *soil deterioration* upon which Gaffney (1965) elaborated. Fertility depletion occurs when the productive capacity of the soil can be restored at a lower cost than the cost of land conservation<sup>1</sup>. By contrast, soil deterioration characterizes a situation where soil properties cannot be restored economically. Soil deterioration leads to a permanent reduction in soil productivity and a decrease in rent and land value (Bunce 1942). The cost of site treatments, such as broadcast fertilization, irrigation, and drainage,

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<sup>1</sup>The cost of conservation is at most the opportunity cost of not cultivating the land, which equals the foregone rent.



are so high in forestry that the restoration of site quality after impairment of the environment is rarely economical. In Bunce's words, what the forester faces is *site deterioration*, characterized by a decreasing site index. Site deterioration is either exogenous, when caused by external forces, or endogenous, when resulting from silvicultural practices.

Stark (1978) presents a case of exogenous site deterioration to develop the concept of the biological life of a soil. Some soils become unable to retain nutrients within the root zone, and nutrients are then leached out. Nutrient leaching is essentially exogenous, contingent upon the physical properties of the soil and precipitation. The duration of the biological life of the soil is determined by how fast the soil loses its productive capacity<sup>1</sup>. Exogenous site deterioration from nutrient leaching commonly occurs in the tropics. It occurs in temperate regions too, on sites with chemically fragile soils.

Kimmins (1974) introduces the concepts of ecological rotation and nutrient recovery period, and provides examples of endogenous site deterioration. At the forest level, the theory of plant succession views each harvest as a disturbance that returns the site to an earlier stage of the sere<sup>2</sup>. The *ecological rotation* is the time necessary for natural succession to recreate the same plant community that was in place before logging. At the soil level,

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<sup>1</sup>Exogenous site deterioration is considered to be relatively slow in this study, so the classic assumption of perpetual tree growth is kept anyway.

<sup>2</sup>The sere is the sequence of plant and animal communities which occupy the site successively.

the removal of timber results in some depletion of the site nutrient capital. The *nutrient recovery period* is the time required for the site to replace lost nutrients through rock weathering and precipitation. If the silvicultural rotation is shorter than either the ecological rotation or the nutrient recovery period, endogenous site deterioration occurs.

Site quality can thus be assumed to change over time and, in particular, to deteriorate. Before the impact of site deterioration on the investment strategies in even-aged forestry is examined, the various models used by the conventional approach to optimum investment in even-aged forestry under conditions of perfect competition will be presented.

### CHAPTER THREE:

#### THE RENT-MAXIMIZATION MODEL OF FULLY REGULATED FORESTS

What is known in forestry as Faustmann's formula was developed by its German author, Martin Faustmann, in 1849 to calculate the rent of forest land under even-aged management (Gane 1968). It can be written as

$$R = \frac{r(pV - ce^{rT})}{e^{rT} - 1},$$

where

R is the annual rent per unit area,

T is the rotation age of the stand in years,

V is the volume of timber per unit area obtained from the stand at age T,

p is the unit price of timber *net of harvesting cost*,

r is the annual interest rate under continuous compounding,

c is the total cost of stand establishment (site preparation and planting),

e is the base of natural logarithms.

The original formula allows for intermediate costs (administration of the stand) and revenues (thinnings) that have been ignored here for simplicity. In addition, the formula has been adjusted to use continuous instead of annual compounding.

On the other hand, the *accrued* rent for one year is an amount slightly greater than  $R$  calculated as follows:

$$R_1 = R \int_0^1 e^{rt} dt = \frac{R(e^r - 1)}{r},$$

where  $R_1$  is the accrued rent at the end of the first year.

Faustmann derived land value (soil expectation value) from rent by using the relationship

$$R = rL,$$

where  $L$  is the land value per unit area. A variation of Faustmann's formula is therefore

$$L = \frac{PV - C e^{rT}}{e^{rT} - 1}.$$

As shown in Appendix A, Faustmann's formula relies on two assumptions which bring limitations to its application. The first and most fundamental assumption is that tree growth is perpetual. Nothing occurs to stop the production of timber on forest land and the number of rotations is infinite. The second assumption is that ecological and economic conditions are permanent. The situation does not change from one rotation to another and the optimal rotation age is the same for all rotations.

With these two assumptions, Faustmann's formula is the core of nearly all economic models used to develop investment strategies for *fully regulated*

forests under conditions of perfect competition. A fully regulated forest is a collection of even-aged stands that are being managed on the same rotation age. Stands of all ages below rotation age are represented by equal acreages (Davis 1966). The concept is the same as those of a normal forest (Chapman and Meyer 1947) and a synchronized forest (Samuelson 1976). Faustmann demonstrated that the land value and the rent per unit area of a fully regulated forest are equal to the land value and the rent per unit area of any of the even-aged stands that compose it (Gane 1968). For most purposes, the economics of fully regulated forests can be studied through the model of a single even-aged stand.

The volume-yield function of even-aged stands can be seen as a multiperiod production function that relates the input and output levels of all periods, or rotations, within the forester's planning horizon. If thinnings and intermediate inputs are disregarded, the economic dynamics of even-aged stands can be captured by a simple point-input-point-output model (Henderson and Quandt 1958). The same point-input-point-output model was used by Fisher (1930) to determine the optimal rotation age in even-aged forestry and by Wicksell (1934) to determine the optimal aging time in wine growing. The Fisherian version of the model is characterized by a single decision variable, rotation age, and a planning horizon limited to one rotation. It does not incorporate Faustmann's formula, which is based on an infinite planning horizon. The objective function to be maximized in the

Fisherian model is the present value of the product of one rotation minus the cost of establishment. The Fisherian model, however, does not yield a satisfactory solution (Gaffney 1957, Hirshleifer 1970).

Another point-input-point-output model with rotation age as the single decision variable has been applied to even-aged forestry by Gaffney (1957), Hirshleifer (1970), and Samuelson (1976). This model enlarges the forester's planning horizon to infinity and incorporates Faustmann's formula. The objective function to be maximized is either rent (Gaffney and Samuelson) or land value (Hirshleifer). Both versions of the model yield the same solution (Samuelson 1976).

The rent-maximization version of the model can be written as

$$\text{Max}_T \left\{ R = \frac{r(PV - Ce^{rT})}{e^{rT} - 1} \right\}, \quad T \geq 0.$$

First-order condition for a maximum (equilibrium):

$$\frac{dR}{dT} = 0 \Leftrightarrow P \frac{dV}{dT} = rPV + R.$$

**Second-order condition for a maximum (strict concavity):**

$$\frac{d^2 R}{dT^2} < 0 \iff \frac{d^2 V}{dV^2} < 0.$$

The calculations that determine these conditions are detailed in Appendix B.

Jackson (1980) has extended the Gaffney-Hirshleifer-Samuelson model by making establishment input a decision variable along with rotation age.

Faustmann's formula then becomes

$$R = \frac{r(PV - wXe^{rT})}{e^{rT} - 1},$$

where

$X$  is the establishment input (site preparation and planting) per unit area,

$w$  is the unit cost of establishment.

The rent-maximization version of the Jackson model can be written as

$$\text{Max}_{X, T} \left\{ R = \frac{r(PV - wXe^{rT})}{e^{rT} - 1} \right\}, \quad X, T \geq 0.$$

**First-order conditions for a maximum (equilibrium):**

$$(1) \quad \frac{\partial R}{\partial X} = 0 \iff P \frac{\partial V}{\partial X} = we^{rT},$$

$$(2) \quad \frac{\partial R}{\partial T} = 0 \iff P \frac{\partial V}{\partial T} = rPV + R.$$

Second-order condition for a maximum (strict concavity):

$$G = \begin{bmatrix} \frac{\partial^2 V}{\partial X^2} & \frac{\partial^2 V}{\partial T \partial X} - r \frac{\partial V}{\partial X} \\ \frac{\partial^2 V}{\partial X \partial T} - r \frac{\partial V}{\partial X} & \frac{\partial^2 V}{\partial T^2} - r \frac{\partial V}{\partial T} \end{bmatrix} \text{ is negative definite.}$$

The calculations that determine these conditions are detailed in Appendix C.

Jackson's decision to make establishment input a decision variable along with rotation age is founded upon the natural trend of understocked stands toward normality. Normality corresponds to that optimal level of stocking where trees fully utilize the site and yet the stand shows no sign of stagnation (Chapman and Meyer 1949). Normal stands are theoretical stands for which volume yield is always at its maximum (relative to age), from establishment to harvest. Such stands do not exist in reality but are used as a reference to determine the understocked, fully-stocked, or overstocked condition of actual stands at a given age. Because trees suffer less competition in understocked stands, relative growth (growth percent) is higher in those stands than in normal stands of the same age. The lower the stocking, the higher the relative growth at a given age (Duerr 1938).

The volume yield of understocked stands will eventually catch up with that of normal stands at some age. Thus the degree of stocking, measured by the ratio of actual to normal yield, varies directly with age. It varies directly with establishment input too, since site preparation and planting determine both the degree of stocking at inception and the rate of the subsequent



approach to normality (the better the initial condition of the stand, the faster the approach). The dual relationship between stocking and age on one hand, and stocking and establishment input on the other hand, is essential to the model. It gives the forester the option to shorten the rotation and spend more funds on regeneration, or to lengthen the rotation and spend less funds on regeneration, while timber yield at the end of the rotation remains the same. Establishment input and rotation age are substitute decision variables (Jackson 1980).

The forestry literature does not provide a simple mathematical expression of the degree of stocking as a function of age and establishment input. Meyer (1933, 1942) and Briegleb (1942) have both studied the *variation* of stocking with age; they have come up with difference equations which cannot be readily solved for the stocking function. Jackson (1980) has worked around Gehrhardt's formula (Duerr 1938) to transform his normal yield function; yet a distinct stocking function cannot be derived from Gehrhardt's formula.

In theory, the degree of stocking of a *naturally regenerated stand* (for which  $X=0$ ) can be expressed as

$$D_0 = 1 - qe^{-bT},$$

where

$D_0$  is the degree of stocking of the naturally regenerated stand at age  $T$ ,

$q$  is the initial stocking with natural regeneration,

$b$  is a positive coefficient.

Hence one possible specification of the stocking of any stand (for which  $X \geq 0$ ) is

$$D = 1 - (qe^{-\gamma X})e^{-bT},$$

where

$D$  is the degree of stocking at age  $T$ ,

$\gamma$  is a positive coefficient.

The unit of establishment input can be chosen so that  $\gamma$  equals 1. The stocking function then becomes

$$D = 1 - qe^{-X-bT}.$$

The volume per acre that can be obtained from any stand is the product of the stocking function by the volume-yield function for normal stands. Its expression is

$$V = D \cdot V_{\infty} = (1 - qe^{-X-bT}) V_{\infty},$$

where  $V_{\infty}$  is the volume per acre at age  $T$  for normal stands corresponding

to an infinite establishment input<sup>1</sup>.

Schumacher (1939) has proposed a volume-yield function for normal stands when volume is measured in *cubic units*. This function is written

$$V_{\infty} = k e^{-\frac{a}{T}},$$

where

k is the maximum volume approached by the stand at a later age,

a is a positive coefficient.

The volume-yield function for any stand becomes

$$V = k (1 - q e^{-X-bT}) e^{-\frac{a}{T}},$$

where V is measured in cubic units.

As shown in Appendix D, this function has the following characteristics over a wide range of ages ( $28 < T < 70$  for loblolly pine, *Pinus taeda*):

---

<sup>1</sup>It is assumed that the volume yield of normal stands is the upper limit of the volume-yield function of actual stands when establishment input tends to infinity.

$$(1) \quad \frac{\partial V}{\partial X} > 0 \quad (2) \quad \frac{\partial V}{\partial T} > 0 \quad (3) \quad \frac{\partial^2 V}{\partial T \partial X} = \frac{\partial^2 V}{\partial X \partial T} < r \frac{\partial V}{\partial X}$$

$$(4) \quad \frac{\partial^2 V}{\partial X^2} < 0 \quad (5) \quad \frac{\partial^2 V}{\partial T^2} < r \frac{\partial V}{\partial T}$$

$$(6) \quad \left[ \frac{\partial^2 V}{\partial T \partial X} - r \frac{\partial V}{\partial X} \right]^2 < \frac{\partial^2 V}{\partial X^2} \left[ \frac{\partial^2 V}{\partial T^2} - r \frac{\partial V}{\partial T} \right]$$

Inequality (3) indicates that establishment input and rotation age are substitute decision variables (Jackson 1980); inequalities (4) and (5) express the law of diminishing returns; inequalities (4), (5), and (6) together ensure that the second-order condition for rent-maximization is satisfied.

The model proposed by Jackson, in which the volume-yield function defined above has been incorporated, can now be used to determine the best investment strategy in even-aged forestry under conditions of perfect competition when site deterioration occurs.

## CHAPTER FOUR:

### RENT MAXIMIZATION WITH EXOGENOUS SITE DETERIORATION

The site index must be included in the rent-maximization model of fully regulated forests before the consequences of site deterioration on the investment strategies in even-aged forestry can be examined. Site deterioration is exogenous if it is caused by external forces only. The site index is then treated as a parameter of the yield function. A famous example of exogenous site deterioration is the leaching of nutrients beyond the roots in tropical soils (Stark 1978). Another example is the rise of the water table on well-drained sites.

Site quality determines the maximum volume approached by the stand at a later age (MacKinney and others 1937). To a lesser extent, site quality also determines relative growth (growth percent), that is, the relative rate at which yield approaches its maximum. Nevertheless, the influence of site quality on relative growth has been neglected in nearly all normal-yield tables, which use anamorphic site-index curves (Bruce 1926, MacKinney and others 1937). Under these conditions, the average growth rate (mean annual increment) and the current growth rate (current annual increment) of timber culminate at the same age for all site indices. The influence of site quality on relative growth is also neglected in this study for simplicity. In any

event, site quality does not affect the degree of stocking (Briegleb 1942). All parameters of the yield function can therefore be considered as independent of site quality, except for the maximum yield. The following linear relationship between the maximum yield and site quality has been proposed by MacKinney and others (1937) for normal stands:

$$k = \alpha s - \beta > 0,$$

where

$s$  is the site index,

$\alpha$  and  $\beta$  are positive coefficients.

A generalized yield function, valid for all site indices, can now be derived from the yield function introduced in Chapter 3 as follows:

$$V = k(1 - qe^{-X-bT})e^{-\frac{a}{T}} = (\alpha s - \beta)(1 - qe^{-X-bT})e^{-\frac{a}{T}}.$$

The classic assumption of permanent ecological and economic conditions can now be relaxed, and exogenous site deterioration can be simulated on the rent-maximization model through *comparative statics*. Comparative statics is a mathematical method used to study changes in the decision variables when a parameter changes (Silberberg 1978). In this case, the site index decreases and comparative statics shows in which direction establishment

input and rotation age should progress to maximize rent. The fundamental assumption of perpetual tree growth is kept. Exogenous site deterioration is assumed to be either sporadic like the rise of the water table, or lasting but relatively slow like nutrient leaching. The site is impaired but does not become exhausted; the forest is still renewable. Given the generalized yield function, the first-order conditions for rent maximization presented in Chapter 3

$$(1) \quad P \frac{\partial V}{\partial X} = w e^{rT},$$

$$(2) \quad P \frac{\partial V}{\partial T} = r P V + R,$$

provide two *implicit* relations in  $X$ ,  $T$ ,  $s$ ,  $p$ ,  $r$ , and  $w$ . When the conventional notation of partial derivatives with subscripts is used, those relations are conveniently written

$$(1) \quad P V_X(X, T, s) = w e^{rT},$$

$$(2) \quad P V_T(X, T, s) = r P V(X, T, s) + R(X, T, s, p, r, w).$$

The second-order condition for rent maximization presented in Chapter 3 stipulates that the Hessian matrix of the second partial derivatives of  $R$  is negative definite, which implies that the determinant of the matrix does not equal zero. This condition allows application of the *implicit function theorem* (Silberberg 1978). From the two implicit relations that constitute the first-

order conditions, the two following *explicit* relations can be derived:

$$X = X^*(s, p, r, w),$$

$$T = T^*(s, p, r, w).$$

The establishment input  $X^*$  and the rotation age  $T^*$  are two functions of  $s$ ,  $p$ ,  $r$ , and  $w$  that *always* keep rent at a maximum. If the site index varies, the replacement of the variables  $X$  and  $T$  with the functions  $X^*$  and  $T^*$  leads to the following identities:

$$(1) \quad pV_X(X^*, T^*, s) \equiv we^{rT^*},$$

$$(2) \quad pV_T(X^*, T^*, s) \equiv r pV(X^*, T^*, s) + R(X^*, T^*, s).$$

These two identities can be differentiated partially with respect to  $s$  using the *chain rule* as follows:

$$(1) \quad p \left( V_{XX} \frac{\partial X^*}{\partial s} + V_{XT} \frac{\partial T^*}{\partial s} + \frac{\partial V_X}{\partial s} \right) \equiv r w e^{rT^*} \frac{\partial T^*}{\partial s}$$

$$\Leftrightarrow pV_{XX} \frac{\partial X^*}{\partial s} + (pV_{XT} - r w e^{rT^*}) \frac{\partial T^*}{\partial s} \equiv -p \frac{\partial V_X}{\partial s}$$

$$\Leftrightarrow V_{XX} \frac{\partial X^*}{\partial s} + (V_{XT} - r V_X) \frac{\partial T^*}{\partial s} \equiv - \frac{\partial V_X}{\partial s},$$



$$(2) \quad P \left( V_{TX} \frac{\partial X^*}{\partial S} + V_{TT} \frac{\partial T^*}{\partial S} + \frac{\partial V_T}{\partial S} \right) \equiv r P \left( V_X \frac{\partial X^*}{\partial S} + V_T \frac{\partial T^*}{\partial S} + \frac{\partial V}{\partial S} \right) + \frac{\partial R}{\partial S}$$

$$\Leftrightarrow \quad (V_{TX} - r V_X) \frac{\partial X^*}{\partial S} + (V_{TT} - r V_T) \frac{\partial T^*}{\partial S} \equiv r \frac{\partial V}{\partial S} - \frac{\partial V_T}{\partial S} + \frac{1}{P} \frac{\partial R}{\partial S}$$

In matrix form, this system of identities appears as

$$\begin{bmatrix} V_{XX} & V_{XT} - r V_X \\ V_{TX} - r V_X & V_{TT} - r V_T \end{bmatrix} \begin{bmatrix} \frac{\partial X^*}{\partial S} \\ \frac{\partial T^*}{\partial S} \end{bmatrix} \equiv G \cdot \begin{bmatrix} \frac{\partial X^*}{\partial S} \\ \frac{\partial T^*}{\partial S} \end{bmatrix} \equiv \begin{bmatrix} -\frac{\partial V_X}{\partial S} \\ r \frac{\partial V}{\partial S} - \frac{\partial V_T}{\partial S} + \frac{1}{P} \frac{\partial R}{\partial S} \end{bmatrix}$$

The solutions obtained by applying *Cramer's rule* are

$$\frac{\partial X^*}{\partial S} \equiv \frac{1}{|G|} \cdot \begin{vmatrix} -\frac{\partial V_X}{\partial S} & V_{XT} - r V_X \\ r \frac{\partial V}{\partial S} - \frac{\partial V_T}{\partial S} + \frac{1}{P} \frac{\partial R}{\partial S} & V_{TT} - r V_T \end{vmatrix} \equiv \frac{|G_1|}{|G|},$$

$$\frac{\partial T^*}{\partial S} \equiv \frac{1}{|G|} \cdot \begin{vmatrix} V_{XX} & -\frac{\partial V_X}{\partial S} \\ V_{TX} - r V_X & r \frac{\partial V}{\partial S} - \frac{\partial V_T}{\partial S} + \frac{1}{P} \frac{\partial R}{\partial S} \end{vmatrix} \equiv \frac{|G_2|}{|G|},$$

where  $|G|$  is the determinant of matrix  $G$ .

As shown in Appendices D and E, the following inequalities hold over a wide range of ages ( $28 < T < 85$  for loblolly pine):

$$(1) \quad V_{XX} < 0,$$

$$(2) \quad V_{TT} - r V_T < 0,$$

$$(3) \quad V_{TX} - r V_X = V_{XT} - r V_X < 0,$$

$$(4) \quad -\frac{\partial V_X}{\partial S} < 0,$$

$$(5) \quad r \frac{\partial V}{\partial S} - \frac{\partial V_T}{\partial S} + \frac{1}{P} \frac{\partial R}{\partial S} > 0.$$

Inequalities (2), (3), (4), and (5) imply

$$|G_1| > 0.$$

Inequalities (1), (3), (4), and (5) imply

$$|G_2| < 0.$$

Since  $|G|$  is positive per the second-order condition for rent maximization, the first partial derivatives of  $X^*$  and  $T^*$  with respect to  $s$  have the following characteristics:

$$\frac{\partial X^*}{\partial S} > 0; \quad \frac{\partial T^*}{\partial S} < 0$$

Assuming exogenous site deterioration, that is, a decrease in the site index caused by external forces, these two inequalities command that the forester

reduce establishment input and augment rotation age while the site deteriorates in order to maximize rent. At first glance, this solution looks counterintuitive because it makes stand management less intensive as the forest becomes less productive. Indeed, intensive management with much effort in stand establishment and short rotations is sometimes advocated for sites that are suspected of deterioration. Yet common sense as well as mathematics tell the forester to do otherwise. Because the site deteriorates, funds spent on stands at inception have lower and lower returns and should be diverted toward more profitable ventures. In addition, as land value diminishes, the opportunity cost of delaying future rotations by growing trees now decreases; rotation age should then be increased to raise the average growth rate of timber (mean annual increment).

Endogenous site deterioration will now be examined to find out whether the same solution of decreasing establishment input and increasing rotation age applies to endogenous site deterioration.

## CHAPTER FIVE:

### LAND-VALUE MAXIMIZATION WITH ENDOGENOUS SITE DETERIORATION

Site deterioration is endogenous if it results from silvicultural practices rather than from external forces. An example of endogenous site deterioration is the depletion of the site nutrient capital from recurrent harvesting when the nutrient recovery period exceeds the silvicultural rotation (Kimmins 1974). After a number of rotations, the site nutrient capital reaches some critical level where it becomes limiting to yield, and site quality starts decreasing. As already mentioned in Chapter 1, a necessary condition for true site deterioration to occur is that the level of fertilization required to maintain or to restore site quality be uneconomical.

Endogenous site deterioration makes the economic model of even-aged forestry dynamic, for the net return of a given rotation depends on the establishment input and rotation age of previous rotations. More precisely, site quality as measured by the site index is a decreasing function of *cumulative* yield. The fundamental assumption of perpetual tree growth is removed at this point. Successive rotations will eventually bring site quality down to a level where trees of the desired species do not regenerate. This shortens the forester's planning horizon to a finite number of rotations. The site is exhaustible, the forest nonrenewable. The number of rotations and

their lengths determine the *life* of the forest, while the limited volume of timber that can be produced from the land over the life of the forest determines the *reserves* of the forest. When the site is exhaustible, there is no perpetual income or rent to be maximized. The forester's next best alternative is to maximize land value, measured by the present value of all net returns occurring until exhaustion.

Problems of exhaustible resources, such as oil and minerals, are normally handled by the use of the calculus of variations (Hotelling 1931) or optimal control theory (Clark 1976). Unfortunately, these techniques cannot be readily applied to even-aged forestry because the forester has no direct control over the current growth rate of timber (current annual increment), and because this rate is discontinuous at the time of harvests. On the other hand, *user-cost analysis*, as applied by Scott (1967) to the economics of mining, can serve as a simple tool for studying the economics of even-aged forestry in the context of endogenous site deterioration. The similarity between mineral fields and nonrenewable forests is intuitive: harvesting timber repeatedly on a deteriorating site comes to "mining" that site.

When the site deteriorates, a series of equal establishment inputs and rotation ages is unlikely to maximize land value. Ecological conditions vary from one rotation to another, and varying conditions command varying values of the decision variables to maximize the objective function. On the other hand, the problem of finding the exact series of different establishment

inputs and rotation ages that maximizes land value is very complex. An alternative approach derived from Scott's work on the economics of mining is to assume first that all establishment inputs and all rotation ages are equal (this reduces the number of decision variables to two), and then try to improve the forester's position by increasing the establishment input or rotation age in some rotations while decreasing it in other rotations. Scott's purpose is to reconsider the assumption that a *constant* rate of output over the years is optimal when dealing with exhaustible resources, a task already undertaken by Gray (1914). Similarly, the objective of this chapter is to challenge the assumption that using the *same* establishment input and rotation age for all rotations is optimal when the forest is nonrenewable. Inclusion of user cost in the maximization model allows determination of the optimal trend of the series of establishment inputs and rotation ages, whether upward or downward.

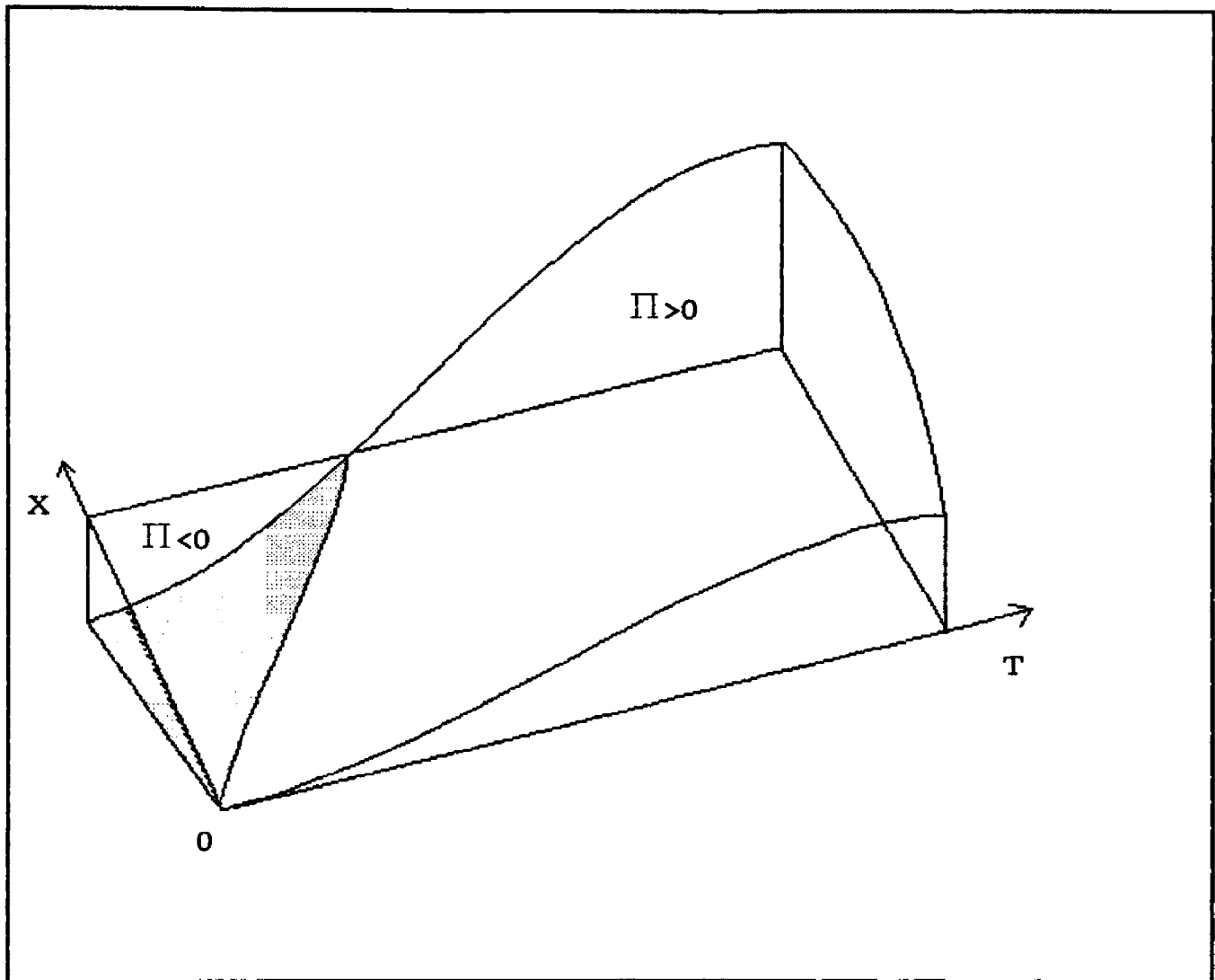
Three assumptions are made so that user-cost analysis can be applied to the economics of even-aged forestry when endogenous site deterioration occurs. First, site deterioration is roughly linear in relation to establishment input and rotation age. Second, the forester knows the reserves of the forest at the beginning of the first rotation. Third, the profit function is identical for all rotations. The third assumption will be removed later in this chapter.

The profit of a rotation is defined as the surplus of the discounted stumpage value over the cost of establishment. This is expressed by

$$\Pi = pVe^{-rT} - wX,$$

where  $\Pi$  is the profit per unit area of the rotation. This equation is actually the objective function of the Fisherian model with two decision variables, characterized by a planning horizon that runs through only one cutting of the forest (Fisher 1930, Hirshleifer 1970). The profit surface of a rotation is illustrated in Figure 1. The *marginal* profit of a rotation is the profit increment realized when establishment input or rotation age is increased by a small quantity. It shows up as the slope of the profit surface for any combination of the decision variables.

Without site deterioration, land value is maximized for values of establishment input and rotation age that are slightly lower than the values of maximum profit per rotation (Gaffney 1957, Samuelson 1976). When the site is exhaustible, however, the optimal values of the decision variables for each rotation are pulled down toward the values of maximum average profit per unit volume of timber (Gray 1914, Scott 1967). The values of establishment input and rotation age of maximum average profit per unit volume of timber and those of maximum profit per rotation define respectively the lower and upper *boundaries of land-value maximization*. Any combination of establishment input and rotation age must lie between these boundaries to maximize land value, regardless of which rotation. Of importance is the concavity of the profit function between the boundaries of land-value maximization. The position of the optimal values of the decision variables



**Figure 1** Profit surface of a rotation.

between the boundaries for each rotation depends on the interest rate and on the reserves of the forest at that point. When the interest rate is low or the reserves are small, land value is maximized near the lower boundary, where the average profit per unit volume of timber is favored. Conversely, when the interest rate is high or the reserves are great, land value is maximized near the upper boundary, where the profit per rotation is favored (Scott 1967).



The concept of user cost developed by Scott (1953) is a generalization of the concept of *prime depreciation* introduced earlier by Lerner (1943). It measures the extra wear and tear of the entrepreneur's assets resulting from a specific activity, to the extent that this wear and tear is not replaced. User cost must be understood as the "cost of use". The user cost of mining or that of growing trees on a deteriorating site is the opportunity cost measured by the present value of the future profit foregone with the decision to mine or to grow trees now (Scott 1967). Present exploitation of an exhaustible resource precludes some future profit by diminishing the reserves and shortening the life of the resource field. The future profit that the forester forfeits by deciding to grow trees in a given rotation is the extra profit he could make in subsequent rotations if, instead, the land lay fallow and the reserves of the forest were saved. The establishment inputs and rotation ages of subsequent rotations could then be increased, or one additional rotation could be scheduled at the end of the exploitation program. In the context of exhaustibility, the forester has to decide whether to allot the limited reserves of the forest to the present (by growing trees) or to the future (by fallowing the land).

Thus the user cost of a rotation is defined in this study as the present value of the extra future profit that would be made if the land lay fallow for the duration of that rotation. It expresses a profit gap, the difference between future profit with fallow and that without fallow. User cost is a

function of establishment input and rotation age. Like the profit function, the user-cost function can be depicted by a surface in a tridimensional space. The *marginal* user cost of a rotation is the present value of the future profit lost when the establishment input or rotation age of that rotation is increased by a small quantity. It is equal to the marginal profit of some future rotation discounted, for some future rotation must sustain a profit cut. When the reserves of the forest are limited, the profit of one rotation can only be increased at the expense of another rotation. The preferred rotation to sustain a profit cut is that with the lowest discounted marginal profit. The marginal user cost shows up as the slope of the user-cost surface for any combination of the decision variables. For values of establishment input and rotation age lying between the boundaries of land-value maximization, the user-cost function is increasing (a greater establishment input or rotation age cuts more deeply into future profit) and convex, inasmuch as the profit function is concave (future marginal profit rises as future profit diminishes). In addition, the level and the slope of the user-cost function keeps rising from one rotation to another because future profit is less discounted.

The first step of user-cost analysis is to make the establishment inputs and rotation ages of all rotations arbitrarily equal and, under this specific condition, to maximize the land value at the beginning of the first rotation. When all establishment inputs and all rotation ages are equal, profit and land

value are related as follows:

$$L_1 = \sum_{k=1}^N \prod e^{-r(k-1)T},$$

where

$L_1$  is the land value per unit area at the beginning of the first rotation,

$N$  is the number of rotations until exhaustion.

With endogenous site deterioration, the number of rotations until exhaustion  $N$  is a function of the establishment input  $X$  and the rotation age  $T$  of all rotations. This function is derived from the implicit relation that ties the reserves of the forest at the beginning of the first rotation (parameter), the establishment input and rotation age of all rotations (independent variables), and the number of rotations until exhaustion (dependant variable).

The decision variables  $X$  and  $T$  are given values that maximize the land value  $L_1$  under the condition of equal establishment inputs and rotation ages. These reference values, written as  $\bar{X}$  and  $\bar{T}$  in this study, are called the "tentatively chosen" establishment input and rotation age in Scott's terminology. Like any optimal combination of the decision variables, they lie between the boundaries of land-value maximization (Scott 1967). When all establishment inputs and rotation ages equal  $\bar{X}$  and  $\bar{T}$  respectively, land

value reaches its *conditional* maximum.

The second step of user-cost analysis is to lift the condition of equal establishment inputs and rotation ages and to adjust the establishment input and rotation age of the first rotation up or down from the reference values  $\bar{X}$  and  $\bar{T}$ . The goal is to increase land value beyond its conditional maximum. The adjustments contemplated are relatively small so as to keep the establishment input and rotation age of the first rotation between the boundaries of land-value maximization; the adjustments are made at the expense or to the benefit of future rotations. The first rotation must meet both a "total" and a "marginal" condition so that land value reaches its *absolute* maximum (Scott 1953, 1967).

The total condition stipulates that profit exceed user cost. Values of establishment input and rotation age to consider are those within a close range of  $\bar{X}$  and  $\bar{T}$ , from which small adjustments are made. If the land lies fallow during the first rotation, most of the extra future profit will come from an additional rotation at the end of the exploitation program, rather than from profit gains in other rotations. The reason is that values of establishment input and rotation age around  $\bar{X}$  and  $\bar{T}$  are more or less optimal for all rotations. Since the profit function has been assumed to be identical for all rotations, the extra future profit will be roughly equal to the profit of the first rotation. User cost, however, is the extra future profit discounted. Therefore, the user cost of the first rotation is less than the

profit, and the total condition is satisfied.

The marginal condition requires that marginal profit equal marginal user cost. As stated previously, the marginal user cost of a rotation is equal to the marginal profit of some future rotation discounted. When all establishment inputs and rotation ages are equal, marginal profit is the same for all rotations. Then marginal user cost equals marginal profit discounted from the *last* rotation on, for the last rotation has the lowest discounted marginal profit. Therefore, the marginal user cost of the first rotation is less than the marginal profit at the tentatively chosen establishment input and rotation age, and the marginal condition is not satisfied. Because the profit function is concave and the user-cost function convex between the boundaries of land-value maximization, both the establishment input and the rotation age of the first rotation must be increased to lower marginal profit and raise marginal user-cost until they are equal. A profit gain must be realized in the first rotation at the expense of subsequent rotations. At this stage, the findings of user-cost analysis can be summarized as follows: with endogenous site deterioration, land-value maximization requires that the establishment input and rotation age of the first rotation exceed those of subsequent rotations.

The two steps of user-cost analysis are then repeated under the conditions in effect at the end of the first rotation. The reserves of the forest are smaller. All establishment inputs and rotation ages are made equal to new

tentatively chosen values at the beginning of the second rotation. These values will be less than the tentatively chosen establishment input and rotation age at the beginning of the first rotation, or the number of rotations until exhaustion will be cut down, because the first rotation has used more than its share of the reserves of the forest. To keep maximizing land value, the establishment input and rotation age of the second rotation must exceed those of subsequent rotations. Furthermore, the optimal values of the decision variables will be somewhat lower for the second than for the first rotation: the slope of the user-cost function being steeper, the marginal condition is satisfied with smaller adjustments of the decision variables.

By going through the same two steps repeatedly, user-cost analysis shows eventually that the establishment input and rotation age of any rotation must be greater than those of the next rotation to maximize land value. The optimal trend of establishment input and rotation ages is downward. The exploitation program must be shifted toward the present, the extent of the shift depending positively on the interest rate and negatively on the concavity of the profit function (Scott 1967). In addition, the number of rotations until exhaustion is less with than without the shift of the exploitation program. As the reserves of the forest are reallocated in favor of the present, the latest rotations are dropped so as to keep the establishment inputs and rotation ages of all remaining rotations above the lower boundary of land-value maximization. This shortens the life of the

forest.

So far in this chapter, the profit function has been assumed to be identical for all rotations. Yet it is more likely to collapse as exhaustion approaches, because the yield of timber falls down. As mentioned in Chapter 4, the level and the slope of the yield function are both tied to site quality. The steady decline of the yield function follows the succession of rotations when site deterioration is endogenous. The assumption of identical profit function can actually be removed without changing the findings of user-cost analysis. With the decline of the profit function, the marginal profit of the latest rotations and hence the marginal user-cost of the earliest rotations drop. Therefore, the establishment input and rotation age of the earliest rotations must be increased even more to satisfy the marginal condition of land value maximization. The shift of the exploitation program toward the present is greater.

Assuming endogenous site deterioration, user-cost analysis shows that the forester must reduce both establishment input and rotation age while the site deteriorates in order to maximize land value. This solution is intuitive in the context of exhaustible resources because the owner of the resource prefers to maximize the average profit per unit output rather than the profit per period<sup>1</sup>. This preference is partially offset by interest which makes future

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<sup>1</sup>The profit per period is maximized when marginal cost equals marginal revenue.

profit less attractive. In the earliest rotations, when the reserves of the forest are great, the discounted value of future stumpage is relatively low and the forester leans more toward maximization of the profit per rotation. In the latest rotations, on the eve of exhaustion, the discounted value of future stumpage is relatively high and the forester leans more toward maximization of the average profit per unit volume of timber.

A conclusion can now be reached on how site deterioration should affect the investment strategies in even-aged forestry, and the economic significance of site deterioration can be assessed.



## CHAPTER SIX:

### CONCLUSION

The consideration of site deterioration in the economics of even-aged forestry brings forth two distinct investment strategies which are contingent upon the cause of deterioration. If the site deteriorates because of ecological forces, site deterioration is exogenous; establishment input must be decreased and rotation age increased from one rotation to another to maximize both rent and land value. If the site deteriorates because of silvicultural practices, site deterioration is endogenous; both establishment input and rotation age must be decreased from one rotation to another to maximize land value. The two investment strategies contrast in the treatment of rotation age. This contrast can be explained by the status of the site in each case with regard to exhaustibility. Ecological forces can impair but will rarely exhaust the site, because their action is either sporadic or slow. On the contrary, detrimental silvicultural practices may exhaust the site in a few rotations. Exhaustibility of the site prompts short rotations, for the average profit per unit volume of timber is favored against the profit per rotation.

Until a new economic model that can handle site deterioration is developed, the timber firm is left with the classic rent or land-value maximization models to optimize establishment input and rotation age in even-aged stands. These

models rely on the assumptions that tree growth is perpetual and that ecological and economic conditions are permanent. If one of the assumptions comes wrong, and in particular if the site deteriorates, the classic models will prescribe values of the establishment input and rotation age that do not maximize rent nor land value. On one hand, establishment input will be too high, and rotation age too small if site deterioration is exogenous. On the other hand, establishment input will be too high, and rotation age too great if site deterioration is endogenous. Consequently, the timber firm will become less competitive and may run into bankruptcy under conditions of perfect competition.

A new economic model with allowance for site deterioration does not guarantee rent or land-value maximization. In addition to the model, the timber firms need periodic measurement of site quality to identify deteriorating sites. The site index does not serve this purpose well, for its determination through stem analysis is too expensive to be more than occasional and topical. Other traditional measurements of site quality, such as habitat types, do not provide any better solution. A new, easier, cheaper method to assess site quality is required before the timber firm can optimize its investment on deteriorating sites. Moreover, some correlation analysis must be made to find out whether site deterioration is endogenous, that is, related to silvicultural practices.

By prompting longer rotations, exogenous site deterioration brings rotation

age closer to the age of maximum average growth rate (mean annual increment). The forest tends toward the state of "maximum sustained yield", treasured by the forester but criticized by the economist (Samuelson 1976). In effect, exogenous site deterioration counterpoises interest by reducing the principal of future returns; it lowers the opportunity cost of delaying future rotations by growing trees now. By contrast, endogenous site deterioration forces the forester to decrease rotation age. The forest is nonrenewable and the state of sustained yield impossible to reach. Yield is essentially not sustainable. Endogenous site deterioration offsets interest too by shortening the life of the forest; it lowers the opportunity cost of precluding future rotations by growing trees now.

## APPENDIX A:

### DERIVATION OF FAUSTMANN'S FORMULA

Although Martin Faustmann primarily intended to determine the land value of normal forests, his method was to calculate rent first and then derive land value (Gane 1968). In the process, he assumed that tree growth was perpetual. Consequently, the number of rotations in Faustmann's formula is infinite, and the forest is expected to generate a continuous income in the form of rent (Chapman 1915). The relation  $R=rL$  ties rent and land value and follows the Ricardian theory, in which rent is an infinite payment stream received by the land owner for the indestructible qualities of the soil. Faustmann further assumed that all ecological and economic conditions were permanent. Hence the optimal rotation age in the formula is the same for all rotations (Clark 1976).

Faustmann used two approaches to derive his rent formula:

(1) The accrued rent of a rotation is the net return at the end of that rotation:

$$R \int_0^T e^{-rt} dt = \frac{R(e^{-rT}-1)}{-r} = PV - Ce^{-rT} \Leftrightarrow R = \frac{r(PV - Ce^{-rT})}{e^{-rT}-1}.$$

(2) Rent is the interest on land value, that is, the present value of all net returns occurring until infinity. Because of the assumption of permanent

conditions, all net returns are identical and their present value amounts to the sum of a geometric series:

$$R = r \sum_{t=1}^{\infty} (pV - ce^{rT}) e^{-rtT} = \frac{r(pV - ce^{rT}) e^{-rT}}{1 - e^{-rT}} = \frac{r(pV - ce^{rT})}{e^{rT} - 1}.$$

In essence, the second approach makes land value the same as the present value of all net returns occurring until infinity, which conforms to modern capital theory:

$$L = \sum_{t=1}^{\infty} (pV - ce^{rT}) e^{-rtT} = \frac{pV - ce^{rT}}{e^{rT} - 1}.$$

## APPENDIX B:

### FIRST- AND SECOND-ORDER CONDITIONS FOR RENT MAXIMIZATION IN THE GAFFNEY-HIRSHLEIFER-SAMUELSON MODEL

Rent function:

$$R = \frac{r}{e^{rT}-1} (PV - C e^{rT}) = \frac{r}{1-e^{-rT}} (PVe^{-rT} - C)$$

$$\begin{aligned} \frac{dR}{dT} = R' &= \frac{r}{1-e^{-rT}} (PV'e^{-rT} - rPVe^{-rT}) - \frac{r^2 e^{-rT}}{(1-e^{-rT})^2} (PVe^{-rT} - C) \\ &= \frac{r e^{-rT}}{1-e^{-rT}} (PV' - rPV - R) = \frac{r}{e^{rT}-1} (PV' - rPV - R) \end{aligned}$$

First-order condition for a maximum (equilibrium):

$$R' = 0 \Leftrightarrow PV' = rPV + R.$$

The marginal interpretation of this condition is that the forester must let trees grow until marginal revenue equals marginal cost. The cost of growing timber is the opportunity cost equal to the interest on the stand value plus the rent on the land value (Gaffney 1957). Martin Faustmann arrived at the same solution but did not express it mathematically (Gane 1968). Instead, he stated that the stand was mature when its *economic* value equalled its *sale* value ( $pV$ ). The economic value of the stand is the forest value ( $pV' + r$ )

minus the land value (L). Hence

$$\frac{pV'}{r} - L = pV \Leftrightarrow pV' = r pV + rL = r pV + R.$$

Assuming  $R' = 0$ , the second derivative of R is calculated as follows:

$$\frac{d^2 R}{dT^2} = R'' = \frac{r}{e^{rT} - 1} (pV'' - r pV' - 0) + 0 = \frac{r p}{e^{rT} - 1} (V'' - r V').$$

Second-order condition for a maximum (strict concavity):

$$R'' < 0 \Leftrightarrow V'' < 0.$$

The second-order condition is met for all ages beyond the age of maximum current annual increment.

## APPENDIX C:

### FIRST- AND SECOND-ORDER CONDITIONS FOR RENT MAXIMIZATION IN THE JACKSON MODEL

**Rent function:**

$$R = \frac{r}{e^{rT}-1} (PV - wXe^{rT}) = \frac{r}{1-e^{-rT}} (PVe^{-rT} - wX)$$

$$\frac{\partial R}{\partial X} = R_X = \frac{r}{e^{rT}-1} (PV_X - we^{rT})$$

$$\begin{aligned} \frac{\partial R}{\partial T} = R_T &= \frac{r}{1-e^{-rT}} (PV_T e^{-rT} - rPVe^{-rT}) - \frac{r^2 e^{-rT}}{(1-e^{-rT})^2} (PVe^{-rT} - wX) \\ &= \frac{r e^{-rT}}{1-e^{-rT}} (PV_T - rPV - R) = \frac{r}{e^{rT}-1} (PV_T - rPV - R) \end{aligned}$$

**First-order conditions for a maximum (equilibrium):**

$$R_X = 0 \Leftrightarrow PV_X = we^{rT},$$

$$R_T = 0 \Leftrightarrow PV_T = rPV + R.$$



Assuming  $R_X = R_T = 0$ , the second partial derivatives of  $R$  are calculated as follows:

$$\frac{\partial^2 R}{\partial X^2} = R_{XX} = \frac{rP}{e^{rT-1}} V_{XX}$$

$$\frac{\partial^2 R}{\partial T \partial X} = R_{XT} = \frac{r}{e^{rT-1}} (P V_{XT} - r W e^{rT}) + 0 = \frac{rP}{e^{rT-1}} (V_{XT} - r V_X)$$

$$\frac{\partial^2 R}{\partial X \partial T} = R_{TX} = \frac{r}{e^{rT-1}} (P V_{TX} - r P V_X - 0) = \frac{rP}{e^{rT-1}} (V_{TX} - r V_X) = R_{XT}$$

$$\frac{\partial^2 R}{\partial T^2} = R_{TT} = \frac{r}{e^{rT-1}} (P V_{TT} - r P V_T - 0) + 0 = \frac{rP}{e^{rT-1}} (V_{TT} - r V_T)$$

Second-order condition for a maximum (strict concavity):

Together with the first order conditions, the sufficient condition for a maximum is that the Hessian matrix of the second partial derivatives of  $R$  be *negative definite* (Silberberg 1978). A matrix is negative definite when all its principal minors of order  $k$  have sign  $(-1)^k$ .

$$H = \begin{bmatrix} R_{XX} & R_{XT} \\ R_{TX} & R_{TT} \end{bmatrix} = \frac{rP}{e^{rT-1}} \begin{bmatrix} V_{XX} & V_{XT} - r V_X \\ V_{TX} - r V_X & V_{TT} - r V_T \end{bmatrix}$$

Let

$$G = \begin{bmatrix} V_{XX} & V_{XT^*} \\ V_{TX^*} & V_{TT^*} \end{bmatrix}$$

**H negative definite  $\Leftrightarrow$  G negative definite.**

**APPENDIX D:**  
**CHARACTERISTICS OF THE VOLUME-YIELD FUNCTION**

**Stocking function:**

$$D = D(X, T) = 1 - qe^{-X-bT}, \quad 1 > D > 0.$$

**Yield function:**

$$V = V(X, T) = kDe^{-\frac{a}{T}}, \quad V > 0.$$

**Partial derivatives of the stocking function:**

$$\frac{\partial D}{\partial X} = D_X = qe^{-X-bT} = 1 - D > 0,$$

$$\frac{\partial D}{\partial T} = D_T = bq e^{-X-bT} = b(1 - D) > 0,$$

$$\frac{\partial^2 D}{\partial T \partial X} = D_{XT} = -bq e^{-X-bT} = b(D - 1) < 0,$$

$$\frac{\partial^2 D}{\partial X \partial T} = D_{TX} = -bq e^{-X-bT} = D_{XT} = b(D - 1) < 0,$$

$$\frac{\partial^2 D}{\partial X^2} = D_{XX} = -qe^{-X-bT} = D - 1 < 0,$$

$$\frac{\partial^2 D}{\partial T^2} = D_{TT} = -b^2 q e^{-X-bT} = b^2(D - 1) < 0.$$

**Partial derivatives of the yield function:**

$$V_X = k D_X e^{-\frac{a}{T}} > 0,$$

$$\begin{aligned} V_T &= k D \frac{a}{T^2} e^{-\frac{a}{T}} + k D_T e^{-\frac{a}{T}} = k \left( \frac{a}{T^2} D + D_T \right) e^{-\frac{a}{T}} \\ &= k \left[ \frac{a}{T^2} D + b(1-D) \right] e^{-\frac{a}{T}} > 0, \end{aligned}$$

$$\begin{aligned} V_{XT} &= k D_X \frac{a}{T^2} e^{-\frac{a}{T}} + k D_{XT} e^{-\frac{a}{T}} \\ &= k \left( \frac{a}{T^2} - b \right) D_X e^{-\frac{a}{T}} = \left( \frac{a}{T^2} - b \right) V_X, \end{aligned}$$

$$\begin{aligned} V_{TX} &= k \left( \frac{a}{T^2} D_X + D_{TX} \right) e^{-\frac{a}{T}} \\ &= k \left( \frac{a}{T^2} - b \right) D_X e^{-\frac{a}{T}} = V_{XT} = \left( \frac{a}{T^2} - b \right) V_X, \end{aligned}$$

$$V_{XX} = k D_{XX} e^{-\frac{a}{T}} = -V_X < 0,$$

$$\begin{aligned} V_{TT} &= k \left( \frac{a}{T^2} D + D_T \right) \frac{a}{T^2} e^{-\frac{a}{T}} + k \left( \frac{-2a}{T^3} D + \frac{a}{T^2} D_T + D_{TT} \right) e^{-\frac{a}{T}} \\ &= k \left[ \frac{a}{T^3} \left( \frac{a}{T} - 2 \right) D + \frac{2a}{T^2} D_T + D_{TT} \right] e^{-\frac{a}{T}} \\ &= k \left[ \frac{a}{T^3} \left( \frac{a}{T} - 2 \right) D + b \left( \frac{2a}{T^2} - b \right) (1-D) \right] e^{-\frac{a}{T}}. \end{aligned}$$

**Sufficient condition for  $V_{XT} = V_{TX} < r V_X$ :**

$$(1) \quad V_{XT} < r V_X \Leftrightarrow \frac{a}{T^2} - b < r \Leftrightarrow (2) \quad T > \sqrt{\frac{a}{b+r}}.$$

In order to specify this condition and those to follow, we need to know the magnitude of the coefficients  $a$  and  $b$ , and of the parameter  $r$ . Schumacher (1939) gives an average value of 37.0 ( $16.0717 \times \log_e 10$ ) to the coefficient  $a$  for loblolly pine (*Pinus taeda*). Meyer (1942) provides the difference equation

$$\Delta D = 0.081 - 0.070D$$

to express the five-year change in stocking for loblolly pine. This equation is very similar to

$$D_T = b - bD.$$

The average value of the coefficient  $b$  for loblolly pine can thus be approximated by 0.015  $[(0.081 + 0.070) \div 2 \div 5]$ . A realistic value for the parameter  $r$  is  $8\% = 0.08$ . Condition (2) becomes  $T > 20$  for loblolly pine.

Sufficient condition for  $V_{TT} - rV_T < 0$ :

$$(3) \quad V_{TT} - rV_T < 0 \Leftrightarrow \frac{a}{T^2} \left( \frac{a}{T^2} - \frac{2}{T} - r \right) D + b \left( \frac{2a}{T^2} - b - r \right) (1 - D) < 0$$

$$\frac{a}{T^2} - \frac{2}{T} - r < 0 \Leftrightarrow a - 2T - rT^2 < 0 \Leftrightarrow (4) \quad T > \frac{-1 + \sqrt{1 + ra}}{r} = 12$$

$$\frac{2a}{T^2} - b - r < 0 \Leftrightarrow T^2 > \frac{2a}{b+r} \Leftrightarrow (5) \quad T > \sqrt{\frac{2a}{b+r}} = 28$$

$$(4) \text{ and } (5) \Rightarrow (3)$$

**Sufficient condition for  $(V_{XT} - rV_X)^2 < V_{XX}(V_{TT} - rV_T)$ :**

$$(6) \quad (V_{XT} - rV_X)^2 < V_{XX}(V_{TT} - rV_T)$$

$$\Leftrightarrow -(\frac{a}{T^2} - b - r)^2 V_X^2 > V_X(V_{TT} - rV_T)$$

$$\Leftrightarrow V_{TT} - rV_T + (\frac{a}{T^2} - b - r)^2 V_X < 0$$

$$\Leftrightarrow \frac{a}{T^2} (\frac{a}{T^2} - \frac{2}{T} - r) D + \left[ \frac{a^2}{T^4} - \frac{2ra}{T^2} + r(b+r) \right] (1-D) < 0$$

$$\Leftrightarrow POLY = a^2 - 2aDT - ra(2-D)T^2 + r(b+r)(1-D)T^4 < 0$$

The following table gives the values of the polynomial expression for  $25 < T < 85$  assuming  $q=0.30$  and given the establishment inputs  $X=0$  and  $X=0.05$ :

T	D(X=0)	POLY(X=0)	D(X=.05)	POLY(X=.05)
25	.794	-1,719	.804	-1,749
35	.823	-3,012	.831	-3,095
45	.847	-3,594	.855	-3,822
55	.869	-3,184	.879	-3,572
65	.887	-1,487	.892	-2,126
70	.895	-134	.900	-1,000
75	.903	1,418	.907	500
85	.916	5,750	.920	4,223

$$(7) \quad 25 < T < 70 \Rightarrow (6)$$

The most restrictive condition among (2), (4), (5), and (7) is  $28 < T < 70$ .

Therefore (1), (3), and (6) are satisfied if  $28 < T < 70$  for loblolly pine.

APPENDIX E:  
CHARACTERISTICS OF THE VOLUME-YIELD AND RENT FUNCTIONS  
RELATIVE TO A CHANGE IN THE SITE INDEX

Volume-yield function:

$$V = (\alpha S - \beta) D e^{-\frac{a}{T}},$$

$$V_X = (\alpha S - \beta) D_X e^{-\frac{a}{T}},$$

$$V_T = (\alpha S - \beta) \left( \frac{a}{T^2} D + D_T \right) e^{-\frac{a}{T}}.$$

Rent function:

$$R = \frac{r(PV - WX e^{rT})}{e^{rT} - 1}.$$



**Partial derivatives with respect to the site index:**

$$\frac{\partial V}{\partial S} = \alpha D e^{-\frac{a}{T}} > 0,$$

$$\frac{\partial V_X}{\partial S} = \alpha D_X e^{-\frac{a}{T}} > 0,$$

$$\frac{\partial V_T}{\partial S} = \alpha \left( \frac{a}{T^2} D + D_T \right) e^{-\frac{a}{T}} > 0,$$

$$\frac{\partial R}{\partial S} = \frac{rP}{e^{rT}-1} \frac{\partial V}{\partial S} > 0,$$

**Sufficient condition for  $\frac{\partial V}{\partial S} + \frac{1}{P} \frac{\partial R}{\partial S} > \frac{\partial V_T}{\partial S}$ :**

$$(1) \quad \frac{\partial V}{\partial S} + \frac{1}{P} \frac{\partial R}{\partial S} > \frac{\partial V_T}{\partial S} \Leftrightarrow \frac{r e^{rT}}{e^{rT}-1} \frac{\partial V}{\partial S} > \frac{\partial V_T}{\partial S}$$

$$\Leftrightarrow \frac{r e^{rT}}{e^{rT}-1} D > \frac{a}{T^2} D + b(1-D)$$

$$\Leftrightarrow POLY = \frac{a}{T^2} D + b(1-D) - \left( \frac{r e^{rT}}{e^{rT}-1} \right) < 0.$$

The following table gives the values of the polynomial expression for  $25 < T < 85$  assuming  $q=0.30$  and given the establishment inputs  $X=0$  and  $X=0.05$ . Values of the stocking function are given in Appendix D.

T	POLY(X=0)	POLY(X=.05)
25	-0.042	-0.042
35	-0.058	-0.058
45	-0.064	-0.064
55	-0.068	-0.068
65	-0.071	-0.071
75	-0.073	-0.073
85	-0.074	-0.074

Therefore (1) is satisfied if  $25 < T < 85$  for loblolly pine.

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